Hausdorff Dimension and Oracle Constructions*

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Abstract

Bennett and Gill (1981) proved that $\mathrm{P}^A \neq \mathrm{NP}^A$ relative to a random oracle A, or in other words, that the set $\mathcal{O}_{[\mathrm{P}=\mathrm{NP}]} = \{A \mid \mathrm{P}^A = \mathrm{NP}^A\}$ has Lebesgue measure 0. In contrast, we show that $\mathcal{O}_{[\mathrm{P}=\mathrm{NP}]}$ has Hausdorff dimension 1.

This follows from a much more general theorem: if there is a *relativizable* and *paddable* oracle construction for a complexity-theoretic statement Φ, then the set of oracles relative to which Φ holds has Hausdorff dimension 1.

We give several other applications including proofs that the polynomial-time hierarchy is infinite relative to a Hausdorff dimension 1 set of oracles and that $\mathrm{P}^A \neq \mathrm{NP}^A \cap \mathrm{coNP}^A$ relative to a Hausdorff dimension 1 set of oracles.

1 Introduction

Bennett and Gill [1] initiated the study of *random oracles* in computational complexity theory. They showed that if an oracle A is chosen uniformly at random, then $\mathrm{P}^A \neq \mathrm{NP}^A$ with probability 1. More precisely, they proved that the set of oracles

$$\mathcal{O}_{[\mathrm{P}=\mathrm{NP}]} = \{A \mid \mathrm{P}^A = \mathrm{NP}^A\}$$

has Lebesgue measure 0.

Hausdorff dimension [7], the most commonly used fractal dimension, provides a quantitative distinction among the measure 0 sets. Every set $\mathcal{O}$ of oracles has a Hausdorff dimension $\dim_{\mathrm{H}}(\mathcal{O})$, a real number in $[0, 1]$. If $\mathcal{O}$ does not have measure 0, then $\dim_{\mathrm{H}}(\mathcal{O}) = 1$, but there are measure 0 sets of each dimension between 0 and 1 inclusive.

It is therefore interesting to ask: what is the Hausdorff dimension of $\mathcal{O}_{[\mathrm{P}=\mathrm{NP}]}$? We prove that

$$\dim_{\mathrm{H}}(\mathcal{O}_{[\mathrm{P}=\mathrm{NP}]}) = 1. \tag{1.1}$$

While $\mathcal{O}_{[\mathrm{P}=\mathrm{NP}]}$ is probabilistically small, there is a dimension-theoretic abundance of oracles A that satisfy $\mathrm{P}^A = \mathrm{NP}^A$.

We establish (1.1) as a corollary of a very general theorem. Let Φ be a relativizable complexity-theoretic statement. In Section 3 we prove that if there is a *paddable* and *relativizable* oracle construction for Φ, then

$$\mathcal{O}_{[\Phi]} = \{A \mid \Phi \text{ holds relative to } A\}$$

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has Hausdorff dimension 1. The proof of this theorem is facilitated by the equivalence of Hausdorff dimension and *log-loss unpredictability* [10].

In Section 4 we give several applications of the general theorem, including (1.1) and that some other measure 0 oracle sets including $\mathcal{O}_{[\mathrm{NP}=\mathrm{EXP}]}$ and $\mathcal{O}_{[\mathrm{P}\neq\mathrm{BPP}]}$ also have Hausdorff dimension 1. It is not known whether $\mathrm{P}^A \neq \mathrm{NP}^A \cap \mathrm{coNP}^A$ relative to a random oracle A or whether the polynomial-time hierarchy has infinitely many distinct levels relative to a random oracle A. We show that each of these statements holds relative to a Hausdorff dimension 1 set of oracles.

2 Dimension and Unpredictability

In this section we review Hausdorff dimension and an equivalent definition of it using log-loss prediction.

Hausdorff dimension is defined in any metric space. In this paper we use the *Cantor space* $\mathbf{C} = \{0,1\}^\infty$ of all infinite binary sequences. As is standard, each oracle $O \subseteq \{0,1\}^*$ is identified with its characteristic sequence $\chi_O \in \mathbf{C}$ according to the lexicographic ordering of $\{0,1\}^*$.

The metric on Cantor space is defined as $\rho(S,T) = 2^{-k}$ where k is the length of longest common prefix of S and T. The *diameter* of a set $Y \subseteq \mathbf{C}$ is $\mathrm{diam}(Y) = \sup\{\rho(S,T) \mid S,T \in Y\}$.

Let $X \subseteq \mathbf{C}$ and $\delta > 0$. We say that a collection $(Y_i)_{i=0}^\infty$ of subsets of $\mathbf{C}$ is a *δ-cover of* X if (i) $\mathrm{diam}(Y_i) \leq \delta$ for all i and (ii) $X \subseteq \bigcup_{i=0}^\infty Y_i$. For each $s \in [0,\infty)$, we define

$$H^s_\delta(X) = \inf\left\{ \sum_{i=0}^{\infty} \mathrm{diam}(Y_i)^s \,\middle|\, (Y_i)_{i=0}^\infty \text{ is a } \delta\text{-cover of } X \right\}.$$

The *s-dimensional Hausdorff outer measure of* X is

$$H^s(X) = \lim_{\delta \to 0} H^s_\delta(X).$$

This limit always exists, but it may be infinite. For each X there is a unique $s^* \in [0,1]$ such that

$$s > s^* \Rightarrow H^s(X) = 0$$

and

$$s < s^* \Rightarrow H^s(X) = \infty.$$

This number s^* is the Hausdorff dimension of X.

Definition. The *Hausdorff dimension* of a set $X \subseteq \mathbf{C}$ is

$$\dim_{\mathrm{H}}(X) = \inf\{s \mid H^s(X) = 0\}.$$

We have $0 \leq \dim_{\mathrm{H}}(X) \leq 1$ for every $X \subseteq \mathbf{C}$. If X does not have Lebesgue measure 0, then $\dim_{\mathrm{H}}(X) = 1$. For each $\alpha \in [0,1]$ there exist sets X with $\dim_{\mathrm{H}}(X) = \alpha$. Hausdorff dimension therefore makes quantitative distinctions among the measure 0 sets. We refer to the book by Falconer [4] for more information about Hausdorff dimension.

We now recall an equivalent definition of Hausdorff dimension involving log-loss prediction [10].

Definition. A *predictor* is a function

$$\pi : \{0,1\}^* \times \{0,1\} \to [0,1]$$

that satisfies

$$\pi(w,0) + \pi(w,1) = 1$$

for all $w \in \{0,1\}^*$.

Intuitively, $\pi(w, b)$ is interpreted as the probability given by the predictor for b following w. The performance of a predictor is measured according to the *log loss* function, a very common loss function in the information theory literature. If probability p was assigned to the outcome that occurred, then the log loss is

$$\log \frac{1}{p}.$$

Definition. Let π be a predictor.

1. The *cumulative log-loss* of π on a string $w \in \{0,1\}^*$ is

$$\mathcal{L}^{\log}(\pi, w) = \sum_{i=0}^{|w|-1} \log \frac{1}{\pi(w \upharpoonright i, w[i])}.$$

2. The *log-loss rate* of π on a sequence $A \in \mathbf{C}$ is

$$\mathcal{L}^{\log}(\pi, A) = \liminf_{n \to \infty} \frac{\mathcal{L}^{\log}(\pi, A \upharpoonright n)}{n}.$$

3. The *worst-case log-loss rate* of π on a set $X \subseteq \mathbf{C}$ is

$$\mathcal{L}^{\log}(\pi, X) = \sup_{A \in X} \mathcal{L}^{\log}(\pi, A).$$

Hausdorff dimension admits an equivalent definition as log-loss unpredictability. Let Π be the set of all predictors. The proof of the following theorem used Lutz's *gale characterization* of Hausdorff dimension [13].

Theorem 2.1. (Hitchcock [10]) *For every* $X \subseteq \mathbf{C}$*,*

$$\dim_{\mathrm{H}}(X) = \inf_{\pi \in \Pi} \mathcal{L}^{\log}(\pi, X).$$

The following lemma can be derived from [13] and [10]; a direct proof is included here for completeness. Intuitively, if π stops making predictions after reading w, it will have loss $\mathcal{L}^{\log}(\pi, wv') = \mathcal{L}^{\log}(\pi, w) + |v'|$. Lemma 2.2 says that the strings $v \in \{0,1\}^l$ on which π can achieve a loss $\log \alpha$ less than this for some prefix of v are at most a $\frac{1}{\alpha}$ fraction of the length l strings.

Lemma 2.2. *Let* π *be a predictor and let* $\alpha > 1$ *be a real number. For all* $l \in \mathbb{N}$ *and* $w \in \{0,1\}^*$*, there are at most* $\frac{2^l}{\alpha}$ *strings* $v \in \{0,1\}^l$ *for which*

$$(\exists v' \sqsubseteq v)\ \mathcal{L}^{\log}(\pi, wv') \leq \mathcal{L}^{\log}(\pi, w) + |v'| - \log \alpha.$$

Proof. Let

$$A = \{v \in \{0,1\}^l \mid (\exists v' \sqsubseteq v) \mathcal{L}^{\log}(\pi, wv') \leq \mathcal{L}^{\log}(\pi, w) + |v'| - \log \alpha\}.$$

Let B be the set of all strings that $v \in \{0,1\}^{\leq l}$ that satisfy $\mathcal{L}^{\log}(\pi, wv) \leq \mathcal{L}^{\log}(\pi, w) + |v| - \log \alpha$ but no prefix of v satisfies this condition. Then $A = \{v \in \{0,1\}^l \mid (\exists v' \sqsubseteq v) v' \in B\}$ and

$$|A| = \sum_{v \in B} 2^{l - |v|} = 2^l \sum_{v \in B} 2^{-|v|}$$

because B is a prefix set. Define a function $\mu : \{0,1\}^{\leq l} \to [0,1]$ by $\mu(\lambda) = 1$ and $\mu(vb) = \mu(v)\pi(wv, b)$ for all $v \in \{0,1\}^{<l}$ and $b \in \{0,1\}$. Then since B is a prefix set, it can be verified that $\sum_{v \in B} \mu(v) \leq 1$. Also, we have $\mu(v) \geq \alpha 2^{-|v|}$ for any $v \in B$ because $\mathcal{L}^{\log}(\pi, wv) - \mathcal{L}^{\log}(\pi, w) = \log \frac{1}{\mu(v)}$. Putting everything together, we have

$$1 \geq \sum_{v \in B} \mu(v) \geq \sum_{v \in B} \alpha 2^{-|v|} = \alpha \frac{|A|}{2^l},$$

so $|A| \leq \frac{2^l}{\alpha}$. $\square$

3 Paddable and Relativizable Oracle Constructions

For each $k \geq 1$, define a padding function $\mathrm{pad}_k : \{0,1\}^* \to \{0,1\}^*$ by

$$\mathrm{pad}_k(x) = 0^{|x|^k - |x|} x$$

and let

$$R_k = \mathrm{range}(\mathrm{pad}_k).$$

Let

$$\mathcal{O}_k = \{B \subseteq \{0,1\}^* \mid B \cap R_k = \emptyset\}$$

be the class of all oracles that are disjoint from R_k.

Definition. Let Φ be a relativizable complexity-theoretic statement. We say that Φ holds via a *paddable and relativizable oracle construction* if

$$(\forall k \geq 1)(\forall B \in \mathcal{O}_k)(\exists A)\ \Phi \text{ holds relative to the oracle } \mathrm{pad}_k(A) \cup B.$$

It seems that most (if not all) oracle constructions for statements Φ involving polynomially bounded computations are paddable and relativizable. First, they are *relativizable* in the sense that for every oracle B there exists an oracle A such that Φ holds relative to the join $A \oplus B = 0A \cup 1B$. Second, they are *paddable* in that if Φ holds relative to A, then Φ also holds relative to $\mathrm{pad}_k(A)$. Here we have combined these two concepts.

We now prove a general theorem that implies many complexity-theoretic statements Φ hold relative to a Hausdorff dimension 1 set of oracles.

Theorem 3.1. *If Φ holds via a paddable and relativizable oracle construction, then*

$$\mathcal{O}_{[\Phi]} = \{A \mid \Phi \text{ holds relative to } A\}$$

has Hausdorff dimension 1.

Proof. Let π be any predictor. By Theorem 2.1, it suffices to show that $\mathcal{L}^{\log}(\pi, \mathcal{O}_{[\Phi]}) \geq 1$.

Let $\epsilon \in (0,1)$. For each $n \in \mathbb{N}$, define $\alpha_n = \lceil 2^{n^\epsilon} \rceil$ and $\beta_n = 2^{\epsilon n}$. Choose n_0 large enough so that $n\alpha_n < \beta_n$ for all $n \geq n_0$.

We will define a sequence of strings v_n for $n \geq 0$ inductively. For $n < n_0$, we let $v_n = 0^{2^n - \alpha_n}$. Now let $n \geq n_0$ and assume that v_i has been defined for all $i < n$. We choose v_n of length $2^n - \alpha_n$ such that for all

$$(u_0, \ldots, u_n) \in \prod_{i=0}^{n} \{0,1\}^{\alpha_i},$$

we have

$$\mathcal{L}^{\log}(\pi, u_0v_0\cdots u_nv'_n) > \mathcal{L}^{\log}(\pi, u_0v_0\cdots u_n) + |v'_n| - \beta_n \tag{3.1}$$

for all $v'_n \sqsubseteq v_n$. Since Lemma 2.2 tells us that for each $(u_0,\ldots,u_n)$ there are at most $2^{2^n-\alpha_n-\beta_n}$ strings $v \in \{0,1\}^{2^n-\alpha_n}$ that satisfy $\mathcal{L}^{\log}(\pi, u_0v_0\cdots u_nv') \leq \mathcal{L}^{\log}(\pi, u_0v_0\cdots u_n) + |v'| - \log 2^{\beta_n}$ for some $v' \sqsubseteq v$ and there are $\prod_{i=0}^{n} 2^{\alpha_i} \leq 2^{n\alpha_n}$ choices of $(u_0,\ldots,u_n)$, we know that such a string v_n exists because

$$2^{n\alpha_n}\cdot 2^{2^n-\alpha_n-\beta_n} < 2^{2^n-\alpha_n}.$$

Let B have the characteristic sequence that is the concatenation of $0^{\alpha_n}v_n$ for all $n\in\mathbb{N}$. In other words, B is empty on the first α_n strings of length n, and the remaining strings are decided according to v_n.

Let $k > \frac{1}{\epsilon}$. We have $B \in \mathcal{O}_k$, so by the hypothesis there is some A such that Φ holds relative to the oracle $C = \mathrm{pad}_k(A)\cup B$.

Let w_n be the length 2^n-1 prefix of C. For any u with $w_nu \sqsubseteq C$ and $|u|\leq\alpha_n$ we have

$$\begin{aligned}\mathcal{L}^{\log}(\pi, w_nu) &\geq \mathcal{L}^{\log}(\pi, w_n)\\ &\geq \mathcal{L}^{\log}(\pi, w_n) + |u| - \alpha_n.\end{aligned}$$

For u, v with $w_nuv \sqsubseteq C$, $|u| = \alpha_n$, and $|v| \leq 2^n - \alpha_n$, we know that

$$\begin{aligned}\mathcal{L}^{\log}(\pi, w_nuv) &> \mathcal{L}^{\log}(\pi, w_nu) + |v| - \beta_n\\ &\geq \mathcal{L}^{\log}(\pi, w_n) + |v| - \beta_n\\ &= \mathcal{L}^{\log}(\pi, w_n) + |uv| - \alpha_n - \beta_n.\end{aligned}$$

Let $m = 2^{n_0}-1$ and let $c = \mathcal{L}^{\log}(\pi, C\upharpoonright m)$. Let w'_n such that $|w'_n| \leq 2^n$ and $w_nw'_n \sqsubseteq C$. We have by induction that

$$\mathcal{L}^{\log}(\pi, w_nw'_n) \geq c + |w_nw'_n| - m - \sum_{i=n_0}^{n}(\alpha_n+\beta_n) \geq |w_nw'_n| - m - n(\alpha_n+\beta_n).$$

It follows that $\mathcal{L}^{\log}(\pi, C) \geq 1$ since m is a constant and $n(\alpha_n+\beta_n) = o(2^n-1)$. Since $C \in \mathcal{O}_{[\Phi]}$, we have $\mathcal{L}^{\log}(\pi, \mathcal{O}_{[\Phi]}) \geq 1$. □

We remark that the proof of Theorem 3.1 can be extended to yield a stronger *scaled dimension* [11] result. It can be shown that the set of oracles has -2^{nd}-order dimension 1.

We conclude this section with a variation of Theorem 3.1 involving random oracles that will be useful in an application. For each $k \geq 1$, let

$$\mathrm{shift}_k : \{0,1\}^* \to R_k^c$$

be the bijection that preserves the lexicographic ordering, where R_k^c is the complement of $R_k = \mathrm{range}(\mathrm{pad}_k)$.

Theorem 3.2. *Suppose that for every $k \geq 1$ there exists an oracle A such that relative to a random oracle R*

$$\Phi \textit{ holds relative to the oracle } \mathrm{pad}_k(A)\cup\mathrm{shift}_k(R) \tag{3.2}$$

with probability 1. Then $\mathcal{O}_{[\Phi]}$ has Hausdorff dimension 1.

Proof. In the proof of Theorem 3.1 we showed that sequence $v_0, v_1, \ldots$ of strings exists by a combinatorial argument. In fact, randomly chosen $v_0, v_1, \ldots$ suffice with high probability. If we choose an oracle R randomly, let $B = \text{shift}_k(R)$, and write $B = w_0 v_0 w_1 v_1 \cdots$ where $|w_n| = \alpha_n$ and $|v_n| = 2^n - \alpha_n$, then with probability 1 the sequence v_0, v_1, ... will satisfy (3.1) for all sufficiently large n. Since (3.2) holds with probability 1, there exists an oracle R with the property of the previous sentence such that (3.2) also holds. Fix such an R. Then Φ holds relative to $C = \text{pad}_k(A) \cup B$ and the rest of the proof goes through to show $\mathcal{L}^{\log}(\pi, C) \geq 1$. □

4 Applications

In this section we apply Theorems 3.1 and 3.2 to some fundamental oracle constructions. We begin with an easy example.

Theorem 4.1. $\mathcal{O}_{[\text{P}=\text{PSPACE}]}$ *has Hausdorff dimension 1.*

Proof. The standard example of an oracle A with $\text{P}^A = \text{PSPACE}^A$ is to let A be PSPACE-complete. We now verify that this is a paddable and relativizable oracle construction.

Let $k \geq 1$ and let $B \in \mathcal{O}_k$. We use

$$K^B = \{\langle x, i, 0^t \rangle \mid M_i^B \text{ accepts } x \text{ in } \leq t \text{ space}\},$$

the canonical PSPACE^B-complete language. Here M_i is the i^{th} oracle Turing machine. Let

$$A = \text{pad}_k(K^B) \cup B.$$

Then A is also PSPACE^B-complete. Since we can directly answer queries to $\text{pad}_k(K^B)$ in polynomial space with access to oracle B, we have $\text{PSPACE}^A = \text{PSPACE}^B$. Therefore

$$\text{P}^A \subseteq \text{PSPACE}^A = \text{PSPACE}^B \subseteq \text{P}^A,$$

so $\text{P}^A = \text{PSPACE}^A$. □

Using the fact that Hausdorff dimension in *monotone*, i.e. $X \subseteq Y$ implies $\dim_{\text{H}}(X) \leq \dim_{\text{H}}(Y)$, the first result mentioned in the introduction follows from Theorem 4.1.

Corollary 4.2. $\mathcal{O}_{[\text{P}=\text{NP}]}$ *has Hausdorff dimension 1.*

Since Bennett and Gill [1] proved that $\text{NP}^A \neq \text{coNP}^A$ relative to a random oracle A, we know that $\mathcal{O}_{[\text{NP}=\text{EXP}]}$ has measure 0. Using Heller's construction of an oracle A with $\text{NP}^A = \text{EXP}^A$ [8], we have a contrasting dimension result.

Theorem 4.3. $\mathcal{O}_{[\text{NP}=\text{EXP}]}$ *has Hausdorff dimension 1.*

Proof. We will show that Heller's oracle construction is paddable and relativizable. Let $k \geq 1$ and let $B \in \mathcal{O}_k$. For any oracle A let $A \oplus_k B = \text{pad}_k(A) \cup B$ and define the language

$$D_k(A, B) = \{\langle i, x, l \rangle \mid M_i^{A \oplus_k B} \text{ accepts } x \text{ in } < l \text{ steps}\}.$$

Then $D_k(A, B)$ is always $\text{EXP}^{A \oplus_k B}$-complete. To apply Theorem 3.1 it suffices to construct an oracle A so that $D_k(A, B) \in \text{NP}^{A \oplus_k B}$. We will construct A to satisfy

$$x \in D_k(A, B) \iff (\exists y)|y| = 3|x| \text{ and } xy \in A$$

for all x. Then $D_k(A,B) \in \mathrm{NP}^A \subseteq \mathrm{NP}^{A\oplus_k B}$.

We construct A in stages. Initially $A = \emptyset$. In stage m, we consider of all x of length m that encode some triple $x = \langle i, a, l\rangle$. We simulate $M_i^{A\oplus_k B}$ on input a for l steps, using the current oracle A. Reserve for A^c all strings $z \notin A$ such that $\mathrm{pad}_k(z)$ is queried in this computation. If $M_i^{A\oplus_k B}$ accepts a in fewer than l steps, we choose some y of length $3m$ such that xy is not reserved for A^c and add xy to A. As argued in [8], we can always choose such a y. This completes stage m. □

The most famous counterexample to the random oracle hypothesis [1] is IP = PSPACE [12, 14, 3]. While IP = PSPACE holds unrelativized, the set $\mathcal{O}_{[\mathrm{IP}=\mathrm{PSPACE}]}$ has measure 0. Since $\mathrm{NP}^A \subseteq \mathrm{IP}^A \subseteq \mathrm{PSPACE}^A \subseteq \mathrm{EXP}^A$ relative to every oracle A, we have the following corollary of Theorem 4.3.

Corollary 4.4. *$\mathcal{O}_{[\mathrm{IP}=\mathrm{PSPACE}]}$ has Hausdorff dimension 1.*

It is not known whether $\mathrm{P}^A \neq \mathrm{NP}^A \cap \mathrm{coNP}^A$ relative to a random oracle A. By the Kolmogorov zero-one law, one of the complementary sets $\mathcal{O}_{[\mathrm{P}=\mathrm{NP}\cap\mathrm{coNP}]}$ and $\mathcal{O}_{[\mathrm{P}\neq\mathrm{NP}\cap\mathrm{coNP}]}$ has measure 1, but it is an open problem to determine which one. From Corollary 4.2, Theorem 4.3, and monotonicity we now know that they both have dimension 1.

Corollary 4.5. *$\mathcal{O}_{[\mathrm{P}=\mathrm{NP}\cap\mathrm{coNP}]}$ and $\mathcal{O}_{[\mathrm{P}\neq\mathrm{NP}\cap\mathrm{coNP}]}$ both have Hausdorff dimension 1.*

Bennett and Gill also showed that $\mathrm{P}^A = \mathrm{BPP}^A$ relative to a random oracle A, or that $\mathcal{O}_{[\mathrm{P}\neq\mathrm{BPP}]}$ has measure 0. Heller [9] constructed an oracle A with $\mathrm{BPP}^A = \mathrm{NEXP}^A$. We can show this oracle construction is paddable and relativizable to establish the following.

Theorem 4.6. *$\mathcal{O}_{[\mathrm{BPP}=\mathrm{NEXP}]}$ has Hausdorff dimension 1.*

Corollary 4.7. *$\mathcal{O}_{[\mathrm{P}\neq\mathrm{BPP}]}$ has Hausdorff dimension 1.*

Yao [15] (see also Håstad [6]) constructed an oracle relative to which the polynomial-time hierarchy has infinitely many distinct levels. Whether this holds relative to a random oracle is an open problem. We now use Theorem 3.2 and a relativized theorem of Book [2, 5] to show that it holds relative to a dimension 1 set of oracles.

Theorem 4.8. *$\mathcal{O}_{[(\forall i)\Sigma_i^{\mathrm{p}}\neq\Sigma_{i+1}^{\mathrm{p}}]}$ has Hausdorff dimension 1.*

Proof. Let A be an oracle such that $\Sigma_i^{\mathrm{p},A} \neq \Sigma_{i+1}^{\mathrm{p},A}$ for all i. By Corollary 3.5 in [5] we know that for a random oracle R, $\Sigma_i^{\mathrm{p},A\oplus R} \neq \Sigma_{i+1}^{\mathrm{p},A\oplus R}$ for all i with probability 1. Noting that $\Sigma_i^{\mathrm{p},A\oplus R} = \Sigma_i^{\mathrm{p},\mathrm{pad}_k(A)\cup\mathrm{shift}_k(R)}$ for every oracle R and $k \geq 1$, we apply Theorem 3.2 and establish the theorem. □

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