Extracting Kolmogorov Complexity with Applications to Dimension Zero-One Laws

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Abstract

We apply results on extracting randomness from independent sources to "extract" Kolmogorov complexity. For any $\alpha, \epsilon > 0$, given a string x with $K(x) > \alpha|x|$, we show how to use a constant number of advice bits to efficiently compute another string y, $|y| = \Omega(|x|)$, with $K(y) > (1-\epsilon)|y|$. This result holds for both unbounded and space-bounded Kolmogorov complexity.

We use the extraction procedure for space-bounded complexity to establish zero-one laws for the strong dimension of complexity classes within ESPACE. We also obtain a similar result for constructive strong dimension.

1 Introduction

Kolmogorov complexity quantifies the amount of randomness in an individual string. If a string x has Kolmogorov complexity m, then x is often said to contain m bits of randomness. Can we efficiently extract the Kolmogorov randomness from a string? That is, given x, is it possible to compute a string of length m that is Kolmogorov-random?

Vereshchagin and Vyugin showed that this is not possible in general [27], i.e., they showed that there is no algorithm that can extract Kolmogorov complexity. Buhrman, Fortnow, Newman and Vereshchagin [4] showed that if one allows a small amount of extra information then Kolmogorov

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extraction is indeed possible. More specifically, they showed there is an efficient procedure $\mathcal{A}$ such that for every x with Kolmogorov complexity αn, there exists a string a_x, such that $\mathcal{A}(x, a_x)$ outputs a nearly Kolmogorov random string whose length is close to αn. Moreover, the length of a_x is $O(\log |x|)$, and contents of a_x depend on x.

In this paper we show that we can extract Kolmogorov complexity with only *constant* bits of additional information. We give a *polynomial-time computable procedure* which takes x with an additional constant amount of advice and outputs a nearly Kolmogorov-random string whose length is linear in m. Formally, for any $\alpha, \epsilon > 0$, given a string x with $K(x) > \alpha|x|$, we show how to use a constant number of advice bits to compute another string y, $|y| = \Omega(|x|)$, in polynomial-time that satisfies $K(y) > (1-\epsilon)|y|$. The number of advice bits depends only on α and ϵ, but the content of the advice depends on x. This computation needs only polynomial time, and yet it extracts unbounded Kolmogorov complexity.

Our proofs use a construction of a *multi-source extractor*. Traditional extractor results [17, 28, 24, 16, 26, 20, 21, 25, 10, 23, 22, 5] show how to take a distribution with high min-entropy and some truly random bits to create a close to uniform distribution. A multi-source extractor takes several independent distributions with high min-entropy and creates a close to uniform distribution. Thus multi-source extractors eliminate the need for a truly random source. Substantial progress has been made recently in the construction of efficient multi-source extractors [2, 3, 19, 18]. In this paper we use the construction due to Barak, Impagliazzo, and Wigderson [2] for our main result on extracting Kolmogorov complexity.

To make the connection, consider the uniform distribution on the set of strings x whose Kolmogorov complexity is at most m. This distribution has min-entropy about m and x acts like a random member of this set. We can define a set of strings $x_1, \ldots, x_k$ to be independent if $K(x_1 \cdots x_k) \approx K(x_1) + \cdots + K(x_k)$. By symmetry of information this implies $K(x_i | x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k) \approx K(x_i)$. Suppose we are given independent Kolmogorov random strings $x_1, \ldots x_k$, whose Kolmogorov complexity is m. We view them as arising from k independent distributions each with min-entropy m. We then argue that a multi-source extractor with small error can be used to output a nearly Kolmogorov random string.

To extract the randomness from a single string x, we break x into a number of substrings $x_1, \ldots, x_l$, and view each substring x_i as coming from a different random source. Of course, these substrings may not be independently random in the Kolmogorov sense, thus we can not view these strings as coming from independent sources. A useful concept is to quantify the *dependency within* x as $\sum_{i=1}^{l} K(x_i) - K(x)$. We show that if the dependency within x is small, then the output of the multi-source extractor on its substrings is a nearly Kolmogorov random string. Another technical problem is that the randomness in x may not be nicely distributed among the substrings; for this we need to use a small (constant) number of nonuniform advice bits.

This result about extracting Kolmogorov-randomness also holds for polynomial-space bounded Kolmogorov complexity. We apply this to obtain zero-one laws for the strong dimensions of certain complexity classes. Resource-bounded dimension and strong dimension [11, 1] were developed as extensions of the classical Hausdorff and packing fractal dimensions to study the structure of complexity classes. Dimension and strong dimension both refine resource-bounded measure and are duals of each other in many ways. Strong dimension is also related to resource-bounded category [8]. In this paper we focus on strong dimension.

The strong dimension of each complexity class is a real number between zero and one inclusive. While there are examples of nonstandard complexity classes with fractional dimensions [1], we do not know of a standard complexity class with this property. Can a natural complexity class have a fractional dimension? In particular consider the class E. Determining its strong dimension within ESPACE would imply a major separation. However, we are able to use our Kolmogorov-randomness extraction procedure to obtain a zero-one law ruling out the intermediate fractional possibility. Formally, we show that the strong dimension Dim(E | ESPACE) is either 0 or 1. The zero-one law also holds for various other complexity classes.

Our techniques also apply in the constructive dimension setting [12]. Miller and Nies [14] asked if it is possible to compute a set of higher constructive dimension from an arbitrary set of positive constructive dimension. We answer the strong dimension variant of this question.

2 Preliminaries

2.1 Kolmogorov Complexity

Let M be a Turing machine. Let $f : \mathbb{N} \to \mathbb{N}$. For any $x \in \{0,1\}^*$, define

$$K_M(x) = \min\{|\pi| \mid M(\pi) \text{ prints } x\}$$

and

$$KS^f_M(x) = \min\{|\pi| \mid M(\pi) \text{ prints } x \text{ using at most } f(|x|) \text{ space}\}.$$

There is a universal machine U such that for every machine M, there is some constant c such that for all x, $K_U(x) \leq K_M(x)+c$ and $KS^{cf+c}_U(x) \leq KS^f_M(x)+c$ [9]. We fix such a machine U and drop the subscript, writing $K(x)$ and $KS^f(x)$, which are called the *(plain) Kolmogorov complexity of* x and f*-bounded (plain) Kolmogorov complexity of* x. While we use plain complexity in this paper, our results also hold for prefix-free complexity.

The following definition quantifies the fraction of randomness in a string.

Definition. For a string x, the *rate* of x is $\mathit{rate}(x) = K(x)/|x|$. For a polynomial g, the g*-rate of* x is $\mathit{rate}^g(x) = KS^g(x)/|x|$.

We denote the uniform distribution over Σ^n with U_n. Two distributions X and Y over Σ^n, are ϵ*-close* if

$$\frac{1}{2}\sum_{x\in\Sigma^n} |X(x) - Y(x)| \leq \epsilon.$$

Definition. Let X be a distribution over Σ^n and $\mathit{Sup}(X)$ denotes the set $\{x \in \Sigma^n \mid \Pr[X = x] \neq 0\}$. The *min-entropy* of X is

$$\min_{x\in \mathit{Sup}(X)} \log \frac{1}{\Pr[X = x]}.$$

2.2 Polynomial-Space Dimension

We now review the definitions of polynomial-space dimension [11] and strong dimension [1]. For more background we refer to these papers and the survey paper [7].

Let $s > 0$. An s*-gale* is a function $d : \{0,1\}^* \to [0,\infty)$ satisfying $2^s d(w) = d(w0) + d(w1)$ for all $w \in \{0,1\}^*$.

For a language A, we write $A\restriction n$ for the first n bits of A's characteristic sequence (according to the standard enumeration of $\{0,1\}^*$) and $A\restriction[i,j]$ for the subsequence beginning from the ith bit and ending at the jth bit. An s-gale d *succeeds on* a language A if $\limsup_{n\to\infty} d(A\upharpoonright n) = \infty$ and d *succeeds strongly on* A if $\liminf_{n\to\infty} d(A\upharpoonright n) = \infty$. The *success set* of d is $S^\infty[d] = \{A \mid d \text{ succeeds on } S\}$. The *strong success set* of d is $S^\infty_{\text{str}}[d] = \{A \mid d \text{ succeeds strongly on } S\}$.

Definition. Let X be a class of languages.

1. The pspace-*dimension* of X is

$$\dim_{\text{pspace}}(X) = \inf\left\{ s \;\middle|\; \begin{array}{l} \text{there is a polynomial-space computable} \\ s\text{-gale } d \text{ such that } X \subseteq S^\infty[d] \end{array} \right\}.$$

2. The *strong* pspace-*dimension* of X is

$$\text{Dim}_{\text{pspace}}(X) = \inf\left\{ s \;\middle|\; \begin{array}{l} \text{there is a polynomial-space computable} \\ s\text{-gale } d \text{ such that } X \subseteq S^\infty_{\text{str}}[d] \end{array} \right\}.$$

For every X, $0 \le \dim_{\text{pspace}}(X) \le \text{Dim}_{\text{pspace}}(X) \le 1$. An important fact is that ESPACE has pspace-dimension 1, which suggests the following definitions.

Definition. Let X be a class of languages.

1. The *dimension of X within* ESPACE is

$$\dim(X \mid \text{ESPACE}) = \dim_{\text{pspace}}(X \cap \text{ESPACE}).$$

2. The *strong dimension of X within* ESPACE is

$$\text{Dim}(X \mid \text{ESPACE}) = \text{Dim}_{\text{pspace}}(X \cap \text{ESPACE}).$$

In this paper we will use an equivalent definition of these dimensions in terms of space-bounded Kolmogorov complexity.

Definition. Given a language L and a polynomial g the *g-rate of L* is

$$rate^g(L) = \liminf_{n\to\infty} rate^g(L\upharpoonright n).$$

strong g-rate of L is

$$Rate^g(L) = \limsup_{n\to\infty} rate^g(L\upharpoonright n).$$

Theorem 2.1. ([13, 6]) *Let* poly *denote all polynomials. For every class X of languages,*

$$\dim_{\text{pspace}}(X) = \inf_{g\in\text{poly}} \sup_{L\in X} rate^g(L).$$

and

$$\text{Dim}_{\text{pspace}}(X) = \inf_{g\in\text{poly}} \sup_{L\in X} Rate^g(L).$$

3 Extracting Kolmogorov Complexity

Barak, Impagliazzo, and Wigderson [2] gave an explicit multi-source extractor.

Theorem 3.1. *([2]) For every constant $0 < \sigma < 1$, and $c > 1$ there exist $l = poly(1/\sigma, c)$, a constant r and a computable function $E : \Sigma^{\ell n} \to \Sigma^n$ such that if $H_1, \cdots, H_l$ are independent distributions over Σ^n, each with min entropy at least σn, then $E(H_1, \cdots, H_l)$ is 2^{-cn}-close to U_n, where U_n is the uniform distribution over Σ^n. Moreover, E runs in time n^r.*

We show that this extractor can be used to produce nearly Kolmogorov-random strings from strings with high enough complexity. The following notion of dependency is useful for quantifying the performance of the extractor.

Definition. Let $x = x_1 x_2 \cdots x_k$, where each x_i is an n-bit string. The *dependency within* x, $dep(x)$, is defined as $\sum_{i=1}^{k} K(x_i) - K(x)$.

Theorem 3.2. *For every $0 < \sigma < 1$ and large enough n, there exist a constant $l > 1$, and a polynomial-time computable function E such that if $x_1, x_2, \cdots x_l$ are n-bit strings with $K(x_i) \geq \sigma n$, $1 \leq i \leq l$, then*

$$K(E(x_1, \cdots, x_l)) \geq n - 10l \log n - dep(x),$$

where $x = x_1 x_2 \cdots x_l$.

Proof. Let $0 < \sigma' < \sigma$. By Theorem 3.1, there is a constant l and a polynomial-time computable multi-source extractor E such that if $H_1, \cdots, H_l$ are independent sources each with min-entropy at least $\sigma' n$, then $E(H_1, \cdots, H_l)$ is 2^{-5n} close to U_n.

We show that this extractor also extracts Kolmogorov complexity. We prove by contradiction. Suppose the conclusion is false, i.e,

$$K(E(x_1, \cdots x_l)) < n - 10l \log n - dep(x).$$

Let $K(x_i) = m_i$, $1 \leq i \leq l$. Define the following sets:

$$I_i = \{y \mid y \in \Sigma^n, K(y) \leq m_i\},$$

$$Z = \{z \in \Sigma^n \mid K(z) < n - 10l \log n - dep(x)\},$$

$$Small = \{\langle y_1, \cdots, y_l \rangle \mid y_i \in I_i, \text{ and } E(y_1, \cdots y_l) \in Z\}.$$

By our assumption $\langle x_1, \cdots x_l \rangle$ belongs to $Small$. We use this to arrive at a contradiction regarding the Kolmogorov complexity of $x = x_1 x_2 \cdots x_l$. We first calculate an upper bound on the size of $Small$.

Observe that the set $\{xy \mid x \in \Sigma^{\sigma' n}, y = 0^{n - \sigma' n}\}$ is a subset of each of I_i. Thus the cardinality of each of I_i is at least $2^{\sigma' n}$. Let H_i be the uniform distribution on I_i. Thus the min-entropy of H_i is at least $\sigma' n$.

Since H_i's have min-entropy at least $\sigma' n$, $E(H_1, \cdots, H_l)$ is 2^{-5n}-close to U_n. Then

$$\Big| P[E(H_1, \ldots, H_l) \in Z] - P[U_n \in Z] \Big| \leq 2^{-5n}. \tag{1}$$

Note that the cardinality of I_i is at most 2^{m_i+1}, as there are at most 2^{m_i+1} strings with Kolmogorov complexity at most m_i. Thus H_i places a weight of at least 2^{-m_i-1} on each string from I_i. Thus $H_1 \times \cdots \times H_l$ places a weight of at least $2^{-(m_1+\cdots+m_l+l)}$ on each element of *Small*. Therefore,

$$P[E(H_1,\ldots,H_l) \in Z] = P[(H_1,\ldots,H_l) \in Small] \geq |Small| \cdot 2^{-(m_1+\cdots+m_l+l)},$$

and since $|Z| \leq 2^{n-10l\log n - dep(x)}$, from (1) we obtain

$$|Small| \quad < \quad 2^{m_1+1} \times \cdots \times 2^{m_l+1} \times \left(\frac{2^{n-10l\log n - dep(x)}}{2^n} + 2^{-5n} \right)$$

Without loss of generality we can take $dep(x) < n$, otherwise the theorem is trivially true. Thus $2^{-5n} < 2^{-10l\log n - dep(x)}$. Using this and the fact that l is a constant independent of n, we obtain

$$|Small| < 2^{m_1+\cdots+m_l - dep(x) - 8l\log n},$$

when n is large enough. Since $K(x) = K(x_1) + \cdots + K(x_l) - dep(x)$,

$$|Small| < 2^{K(x) - 8l\log n}.$$

We first observe that there is a program Q that, given the values of m_i's, n, l, and $dep(x)$ as auxiliary inputs, recognizes the set *Small*. This program works as follows: Let $z = z_1 \cdots z_l$, where $|z_i| = n$. For each program P_i of length at most m_i check whether P_i outputs z_i, by running the P_i's in a dovetail fashion. If it is discovered that for each of z_i, $K(z_i) \leq m_i$, then compute $y = E(z_1, \cdots, z_l)$. Now verify that $K(y)$ is at most $n - dep(x) - 10l\log n$. This again can be done by running programs of the length at most $n - dep(x) - 10l\log n$ in a dovetail manner. If it is discovered that $K(y)$ is at most $n - dep(x) - 10l\log n$, then accept z.

So given the values of parameters n, $dep(x)$, l and m_is, there is a program P that enumerates all elements of *Small*. Since by our assumption x belongs to *Small*, x appears in this enumeration. Let i be the position of x in this enumeration. Since $|Small|$ is at most $2^{K(x)-8l\log n}$, i can be described using $K(x) - 8l\log n$ bits.

Thus there is a program P' based on P that outputs x. This program takes i, $dep(x)$, n, $m_1, \cdots, m_l$, and l, as auxiliary inputs. Since the m_i's and $dep(x)$ are bounded by n,

$$\begin{aligned} K(x) &\leq K(x) - 8l\log n + 2\log n + l\log n + O(1) \\ &\leq K(x) - 5l\log n + O(1), \end{aligned}$$

which is a contradiction. □

If $x_1, \cdots x_l$ are independent strings with $K(x_i) \geq \sigma n$, then $E(x_1, \cdots, x_l)$ is a Kolmogorov random string of length n.

Corollary 3.3. *For every constant $0 < \sigma < 1$, there exists a constant l, and a polynomial-time computable function E such that if $x_1, \cdots x_l$ are n-bit strings such $K(x_i) \geq \sigma n$, and $K(x_1 x_2 \cdots x_l) = \sum K(x_i) - O(\log n)$, then $E(x_1, \cdots, x_l)$ is Kolmogorov random, i.e.,*

$$K(E(x_1, \cdots, x_l)) > n - O(\log n).$$

This theorem says that given $x \in \Sigma^{ln}$, if each piece x_i has high enough complexity and the dependency with x is small, then we can output a string y whose Kolmogorov rate is higher than the Kolmogorov rate of x, i.e, y is relatively more random than x. What if we only knew that x has high enough complexity but knew nothing about the complexity of individual pieces or the dependency within x? Our next theorem states that in this case also there is a procedure producing a string whose rate is higher than the rate of x. However, this procedure needs constant bits of advice.

Theorem 3.4. *For all real numbers $0 < \alpha < \beta < 1$ there exist a constant $0 < \gamma < 1$, constants $c, l, n_0 \geq 1$, and a procedure R such that the following holds. For any string x with $|x| \geq n_0$ and $rate(x) \geq \alpha$, there exists an advice string a_x such that*

$$rate(R(x, a_x)) \geq \min\{rate(x) + \gamma, \beta\}$$

where $|a_x| = c$. Moreover, R runs in polynomial time, and $|R(x, a_x)| = \lfloor |x|/l \rfloor$.

The number c depends only on α, β and is independent of x. However, the contents of a_x depend on x.

Proof. Let $\alpha' < \alpha$ and $\epsilon < \min\{1-\beta, \alpha'\}$. Let $\sigma = (1-\epsilon)\alpha'$. Using parameter σ in Theorem 3.2, we obtain a constant $l > 1$ and a polynomial-time computable function E that extracts Kolmogorov complexity.

Let $\beta' = 1 - \frac{\epsilon}{2}$, and $\gamma = \frac{\epsilon^2}{2l}$. Observe that $\gamma \leq \frac{1-\beta'}{l}$ and $\gamma < \frac{\alpha'-\sigma}{l}$.

Let x have $rate(x) = \nu \geq \alpha$. Let $n, k \geq 0$ such that $|x| = ln + k$ and $k < l$. We strip the last k bits from x and write $x = x_1 \cdots x_l$ where each $|x_i| = n$. Let $\nu' = rate(x)$ after this change. We have $\nu' > \nu - \gamma/2$ and $\nu' > \alpha'$ if $|x|$ is sufficiently large.

We consider three cases.

Case 1. There exists j, $1 \leq j \leq l$ such that $K(x_j) < \sigma n$.
Case 2. Case 1 does not hold and $dep(x) \geq \gamma ln$.
Case 3. Case 1 does not hold and $dep(x) < \gamma ln$.

We have two claims about Cases 1 and 2:

Claim 3.4.1. *Assume Case 1 holds. There exists i, $1 \leq i \leq l$, such that $rate(x_i) \geq \nu' + \gamma$.*

Proof of Claim 3.4.1. Suppose not. Then for every $i \neq j$, $1 \leq i \leq l$, $K(x_i) \leq (\nu' + \gamma)n$. We can describe x by describing x_j which takes σn bits, and all the x_i's, $i \neq j$. Thus the total complexity of x would be at most

$$(\nu' + \gamma)(l - 1)n + \sigma n + O(\log n)$$

Since $\gamma < \frac{\alpha'-\sigma}{l}$ and $\alpha' < \nu'$ this quantity is less than $\nu' ln$. Since the rate of x is ν', this is a contradiction. $\square$ *Claim 3.4.1*

Claim 3.4.2. *Assume Case 2 holds. There exists i, $1 \leq i \leq l$, $rate(x_i) \geq \nu' + \gamma$.*

Proof of Claim 3.4.2. By definition,

$$K(x) = \sum_{i=1}^{l} K(x_i) - dep(x)$$

Since $dep(x) \geq \gamma ln$ and $K(x) \geq \nu' ln$,

$$\sum_{i=1}^{l} K(x_i) \geq (\nu' + \gamma) ln.$$

Thus there exists i such that $rate(x_i) \geq \nu' + \gamma$. $\square$ *Claim 3.4.2*

We can now describe the constant number of advice bits. The advice a_x contains the following information: which of the three cases described above holds, and

- If Case 1 holds, then from Claim 3.4.1 the index i such that $rate(x_i) \geq \nu' + \gamma$.
- If Case 2 holds, then from Claim 3.4.2 the index i such that $rate(x_i) \geq \nu' + \gamma$.

Since $1 \leq i \leq l$, the number of advice bits is bounded by $O(\log l)$. We now describe procedure R. When R takes an input x, it first examines the advice a_x. If Case 1 or Case 2 holds, then R simply outputs x_i. Otherwise, Case 3 holds, and R outputs $E(x)$. Since E runs in polynomial time, R runs in polynomial time.

If Case 1 or Case 2 holds, then

$$rate(R(x, a_x)) \geq \nu' + \gamma \geq \nu + \tfrac{\gamma}{2}.$$

If Case 3 holds, we have $R(x, a_x) = E(x)$ and by Theorem 3.2, $K(E(x)) \geq n - 10 \log n - \gamma ln$. Since $\gamma \leq \frac{1-\beta'}{l}$, in this case

$$rate(R(x, a_x)) \geq \beta' - \tfrac{10 \log n}{n}.$$

For large enough n, this value is at least β. Therefore in all three cases, the rate increases by at least $\gamma/2$ or reaches β. $\square$

We now prove our main theorem.

Theorem 3.5. *Let α and β be constants with $0 < \alpha < \beta < 1$. There exist a polynomial-time procedure $P(\cdot, \cdot)$ and constants C_1, C_2, n_1 such that for every x with $|x| \geq n_1$ and $rate(x) \geq \alpha$ there exists a string a_x with $|a_x| = C_1$ such that*

$$rate(P(x, a_x)) \geq \beta$$

and $|P(x, a_x)| \geq |x|/C_2$.

Proof. We apply the procedure R from Theorem 3.4 iteratively. Each application of R outputs a string whose rate is at least β or is at least γ more than the rate of the input string. Applying R at most $k = \lceil (\beta - \alpha)/\gamma \rceil$ times, we obtain a string whose rate is at least β.

Note that $R(y, a_y)$ has output length $|R(y, a_y)| = \lfloor |y|/l \rfloor$ and increases the rate of y if $|y| \geq n_0$. If we take $n_1 = (n_0 + 1)kl$, we ensure that in each application of R we have a string whose length is at least n_0. Each iteration of R requires c bits of advice, so the total number of advice bits needed is $C_1 = kc$. Thus C_1 depends only on α and β. Each application of R decreases the length by a constant fraction, so there is a constant C_2 such that the length of the final outputs string is at least $|x|/C_2$. $\square$

The proofs in this section also work for space-bounded Kolmogorov complexity. For this we need a space-bounded version of dependency.

Definition. Let $x = x_1 x_2 \cdots x_k$ where each x_i is an n-bit string, let f and g be two space bounds. The (f,g)*-bounded dependency within* x, $dep_g^f(x)$, is defined as $\sum_{i=1}^{k} KS^g(x_i) - KS^f(x)$.

We obtain the following version of Theorem 3.2.

Theorem 3.6. *For every polynomial g there exists a polynomial f such that for every $0 < \sigma < 1$, there exist a constant $l > 1$, and a polynomial-time computable function E such that if $x_1, \cdots, x_l$ are n-bit strings with $KS^f(x_i) \geq \sigma n$, $1 \leq i \leq l$, then*

$$KS^g(E(x_1, \cdots, x_l)) \geq n - 10l \log n - dep_g^f(x).$$

Similarly we obtain the following extension of Theorem 3.5.

Theorem 3.7. *Let g be a polynomial and let α and β be constants with $0 < \alpha < \beta < 1$. There exist a polynomial f, polynomial-time procedure $R(\cdot,\cdot)$, and constants C_1, C_2, n_1 such that for every x with $|x| \geq n_1$ and $rate^f(x) \geq \alpha$ there exists a string a_x with $|a_x| = C_1$ such that*

$$rate^g(R(x, a_x)) \geq \beta$$

and $|R(x, a_x)| \geq |x|/C_2$.

4 Zero-One Laws for Complexity Classes

In this section we establish a zero-one law for the strong dimensions of certain complexity classes.

Lemma 4.1. *Let g be any polynomial and α, θ be rational numbers with $0 < \alpha < \theta < 1$. Then there is a polynomial f such that if there exists $L \in \mathrm{E}$ with $Rate^f(L) \geq \alpha$, then there exists $L' \in \mathrm{E}$ with $Rate^g(L') \geq \theta$.*

Proof. Let β be a real number bigger than θ and smaller than 1 and $f = \omega(g)$. Pick positive integers C and K such that $(C-1)/K < 3\alpha/4$, and $\frac{(C-1)\beta}{C} > \theta$. Let $n_1 = 1$, $n_{i+1} = Cn_i$.

We now define strings $y_1, y_2, \cdots$ such that each y_i is a substring of the characteristic sequence of L or is in 0^*, and $|y_i| = (C-1)n_i/K$. While defining these strings we will ensure that for infinitely many i, $rate^f(y_i) \geq \alpha/4$.

We now define y_i. We consider three cases.

Case 1. $rate^f(L \restriction n_i) \geq \alpha/4$. Divide $L \restriction n_i$ in to $K/(C-1)$ segments such that the length of each segment is $(C-1)n_i/K$. It is easy to see that at least for one segment the f-rate is at least $\alpha/4$. Define y_i to be a segment with $rate^f(y_i) \geq \alpha/4$.

Case 2. Case 1 does not hold and for every j, $n_i < j < n_{i+1}$, $rate^f(L \restriction j) < \alpha$. In this case we punt and define $y_i = 0^{\frac{(C-1)n_i}{K}}$.

Case 3. Case 1 does not hold and there exists j, $n_i < j < n_{i+1}$ such that $rate^f(L \restriction j) > \alpha$. Divide $L \restriction [n_i, n_{i+1}]$ into K segments. Since $n_{i+1} = Cn_i$, length of each segment is $(C-1)n_i/K$.

Then it is easy to show that some segment has f-rate at least $\alpha/4$. We define y_i to be this segment.

Since for infinitely many j, $rate^f(L\upharpoonright j) \geq \alpha$, for infinitely many i either Case 1 or Case 3 holds. Thus for infinitely many i, $rate^f(y_i) \geq \alpha/4$.

By Theorem 3.7, there is a procedure R with such that given a string x with $rate^f(x) \geq \alpha/4$, and the advice a_x, $rate^g(R(x, a_x)) \geq \beta$.

Let $w_i = R(y_i, a_{y_i})$. Since for infinitely many i, $rate^f(y_i) \geq \alpha/4$, for infinitely many i, $rate^g(w_i) \geq \beta$. Also recall that $|w_i| = |y_i|/C_2$ for an absolute constant C_2.

Claim 4.1.1. $|w_{i+1}| \geq (C-1)\sum_{j=1}^{i} |w_j|$.

Proof of Claim 4.1.1. We have

$$\sum_{j=1}^{i} |w_j| \leq \frac{C-1}{KC_2}\sum_{j=1}^{i} n_j = \frac{C-1}{KC_2}\frac{(C^i-1)n_1}{C-1},$$

with the equality holding because $n_{j+1} = Cn_j$. Also,

$$|w_{i+1}| = \frac{(C-1)n_{i+1}}{KC_2} \geq \frac{(C-1)C^i n_1}{KC_2}$$

Thus

$$\frac{|w_{i+1}|}{\sum_{j=1}^{i} |w_j|} > (C-1).$$

□ *Claim 4.1.1*

Claim 4.1.2. *For infinitely many i, $rate^g(w_1 \cdots w_i) \geq \theta$.*

Proof of Claim 4.1.2. For infinitely many i, $rate^g(w_i) \geq \beta$, which means $KS^g(w_i) \geq \beta|w_i|$ and therefore

$$KS^g(w_1 \cdots w_i) \geq \beta|w_i| - O(1).$$

By Claim 4.1.1, $|w_i| \geq (C-1)(|w_1| + \cdots + |w_{i-1}|)$. Thus for infinitely many i, $rate^g(w_1 \cdots w_i) \geq \frac{(C-1)\beta}{C} - o(1) \geq \theta$. □ *Claim 4.1.2*

We define $w_1 w_2 \cdots$ to be the characteristic sequence of L'. Then by Claim 4.1.2, $Rate^g(L') \geq \theta$.

Next, we argue that if L is in E, then L' is in E$/O(1)$. Observe that w_i depends on y_i and a_{y_i}, thus each bit of w_i can be computed by knowing y_i and a_{y_i}. Recall that y_i is either a subsegment of the characteristic sequence of L or 0^{n_i}. We will know y_i if we know which of the three cases mentioned above hold. This can be given as advice. Also observe that y_i is a subsequence of $L \upharpoonright n_{i+1}$. Also recall that w_i can be computed from y_i in time polynomial in $|y_i|$ using constant bits of advice a_{y_i}. Since $|w_i| = |y_i|/C_2$ for some absolute constant C_2, the running time needed to compute w_i is also polynomial in $|w_i|$. Since L is in E, this places L' in E$/O(1)$.

Finally, we observe that the advice can be removed to obtain a language in E. Let I be the set of all i such that $rate^g(w_1 \cdots w_i) \geq \theta$. Let A be the set of all advice strings that are used in computing w_i from $L\upharpoonright n_{i+1}$ for $i \in I$. Since I is infinite and A is finite, there must be some advice string $a \in A$ that can be used to compute infinitely many of the w_i's. We hardcode a into the algorithm for computing L'. Call the new language we get L''. We have $L'' \in$ E. Infinitely often, L'' will be the same as L' on a w_i stretch, and it can be different elsewhere. Observe that in the proof of Claim 4.1.2 changing the strings $w_1, \ldots, w_{i-1}$ has no effect. It follows that $Rate^g(L'') \geq \theta$. This completes the proof of Lemma 4.1. □

Theorem 4.2. $\mathrm{Dim}(\mathrm{E} \mid \mathrm{ESPACE})$ *is either 0 or 1.*

Proof. Because $\mathrm{E} \subseteq \mathrm{ESPACE}$, $\mathrm{Dim}(\mathrm{E} \mid \mathrm{ESPACE}) = \mathrm{Dim}_{\mathrm{pspace}}(\mathrm{E})$. We will show that if $\mathrm{Dim}_{\mathrm{pspace}}(\mathrm{E}) > 0$, then $\mathrm{Dim}_{\mathrm{pspace}}(\mathrm{E}) = 1$. For this, it suffices to show that for every polynomial g and real number $0 < \theta < 1$, there is a language L' in E with $\mathit{Rate}^g(L') \geq \theta$. By Theorem 2.1, this will show that the strong pspace-dimension of E is 1.

The assumption states that the strong pspace-dimension of E is greater than 0. If the strong pspace-dimension of E is actually one, then we are done. If not, let α be a positive rational number that is less than $\mathrm{Dim}_{\mathrm{pspace}}(\mathrm{E})$. By Theorem 2.1, for every polynomial f, there exists a language $L \in \mathrm{E}$ with $\mathit{Rate}^f(L) \geq \alpha$.

By Lemma 4.1, from such a language L we obtain a language L' in E with $\mathit{Rate}^g(L') \geq \theta$. Thus the strong pspace-dimension of E is 1. □

The zero-one law in Theorem 4.2 also holds for many other complexity classes.

Theorem 4.3. *Let $\mathcal{C}$ be a class that is closed under exponential-time truth-table reductions. Then* $\mathrm{Dim}(\mathcal{C} \mid \mathrm{ESPACE})$ *is either 0 or 1.*

Therefore additional examples of classes the zero-one law holds for include $\mathrm{NE} \cap \mathrm{coNE}$, BPE, and E^{NP}.

Remark. Theorem 4.2 concerns strong dimension. For dimension, the situation is considerably more complicated. With our techniques we can prove that if $\dim_{\mathrm{pspace}}(\mathrm{E}) > 0$, then $\dim_{\mathrm{pspace}}(\mathrm{E}/O(1)) \geq 1/2$. It appears that a different method is needed to eliminate the advice or increase the dimension past 1/2.

5 Increasing Constructive Strong Dimension

Miller and Nies [14] asked if every set of positive constructive dimension computes (by way of a Turing reduction) a set of higher constructive dimension. Our techniques yield a positive answer for the variant of this question using strong dimension instead of dimension. For a set S, the constructive strong dimension [1] of S is defined by

$$\mathrm{Dim}(S) = \limsup_{n \to \infty} \frac{K(S \upharpoonright n)}{n}.$$

Theorem 5.1. *If* $\mathrm{Dim}(S) > 0$*, then for every* $\epsilon > 0$*, there exists* $R \leq_{\mathrm{T}} S$ *such that* $\mathrm{Dim}(R) > 1 - \epsilon$.

The proof of Theorem 5.1 is the same as Lemma 4.1, except instead of Theorem 3.7 we use Theorem 3.5. The reduction we obtain is actually an exponential-time truth-table reduction, so in particular it is a weak truth-table reduction. In contrast, Nies and Reimann [15] showed that this is sometimes impossible for constructive dimension: there exists S with $\dim(S) > 0$ such that every set which weak truth-table reduces to S has $\dim(R) \leq \dim(S)$.

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