Base Invariance of Feasible Dimension

John M. Hitchcock[*] Elvira Mayordomo[†]

April 26, 2013

Abstract

Effective fractal dimensions were introduced by Lutz (2003) in order to study the dimensions of individual sequences and quantitatively analyze the structure of complexity classes. Interesting connections of effective dimensions with information theory were also found, implying that constructive dimension as well as polynomial-space dimension are invariant under base-change while finite-state dimension is not.

We consider the intermediate case, polynomial-time dimension, and prove that it is indeed invariant under base-change by a nontrivial argument which is quite different from the Kolmogorov complexity ones used in the other cases.

Polynomial-time dimension can be characterized in terms of prediction-loss-rate, entropy, and compression algorithms. Our result implies that in an asymptotic way each of these concepts is invariant under base-change.

A corollary of the main theorem is any polynomial-time dimension 1 number (which may be established in any base) is an absolutely normal number, providing an interesting source of absolute normality.

1 Introduction

Effective fractal dimensions were introduced by Lutz in order to quantitatively analyze the structure of complexity classes [17] and study the dimensions of individual sequences [18]. Polynomial-time dimension, or feasible dimension, is the polynomial-time effectivization of Hausdorff dimension [17] in the space of infinite sequences over a finite alphabet. Important applications in computational complexity have been found including circuit-size complexity, polynomial-time degrees, the size of NP, zero-one laws, and oracle classes. See [19, 11, 9] for a summary of the main results on effective dimensions. In this paper we prove that polynomial-time dimension is invariant under base change, that is, for different bases the polynomial-time dimension of sequences corresponding to the same set of real numbers coincides.

The concept of randomness of a real number can be naturally defined from the randomness of the binary infinite sequence that represents this number. The choice of base two representation here is an arbitrary one–base three, base four, or any other base would work just as well–they

[*]Department of Computer Science, University of Wyoming, jhitchco@cs.uwyo.edu. This research was supported in part by NSF grants 9988483 and 0917417.

[†]Dept. de Informática e Ingeniería de Sistemas, Instituto de Investigación en Ingeniería de Aragón (I3A), Universidad de Zaragoza, Zaragoza, SPAIN. elvira@unizar.es. Part of this work was done during a visit to the Isaac Newton Institute for Mathematical Sciences at the University of Cambridge. This work was supported by the Spanish Ministry of Science and Innovation (Projects TIN2008-06582-C03-02, TIN2011-27479-C04-01)

all yield the same randomness notions [7, 6]. Surprisingly, when looking at effective versions of randomness one choice of base may not be equivalent to other base representations. For instance, in finite-state randomness and dimension [8] it is known that both randomness and dimension 1 sequences coincide with the normal sequences (consequence of [5, 21]), and therefore both finite-state randomness and dimension are not invariant under base change, since the existence of non-absolutely normal sequences is known [20]. On the other hand, Martin-Löf randomness [9, 7] and constructive dimension [18] can be easily proven to be base-invariant by standard Kolmogorov complexity arguments, and for the same reason pspace-randomness [16] and pspace-dimension [17] are base invariant.

We study an intermediate case, polynomial-time resource-bounds, and prove that polynomial-time dimension (p-dimension) is invariant under base change. The proof is nontrivial since base change is not an honest function (defined in [2]), in fact for infinitely many cases it is arbitrarily length-decreasing. Consider for instance the process of changing the real number 0.5 (in decimal notation) from base 3 to base 2. When given successively longer finite prefixes of the infinite base 3 representation 0.1111..., there are always two possible candidates for finite prefixes of a base 2 representation. This makes any (time-bounded) randomness argument more complicated, while the Kolmogorov complexity of both candidates is very close. Very recently p-randomness has been proven base-invariant [22], a result incomparable to ours.

p-dimension can be characterized in terms of prediction loss rate [10], entropy [13], and compression algorithms [15]. Our result implies that in an asymptotic way each of those concepts is invariant under base-change.

Another consequence of our main result is that p-dimension 1 numbers are absolutely normal, thus providing an interesting source of absolute normality.

Strong-p-dimension [1], a concept dual to p-dimension that is the effectivization of packing dimension from fractal geometry is also base-invariant, which can be proven with an argument similar to that used in the proof of our main theorem.

2 Preliminaries

2.1 p-dimension

For any natural number $k \geq 2$, we let $\Sigma_k = \{0, \dots, k-1\}$ be a k-symbol alphabet. Σ_k^* denotes the set of finite strings over alphabet Σ_k, Σ_k^∞ denotes infinite sequences over alphabet Σ_k.

For $0 \leq i \leq j$, we write $x[i \dots j]$ for the string consisting of the i-th through the j-th symbols of x. We use λ for the empty string.

Definition. Let $s \in [0, \infty)$. An *s-gale* on Σ_k is a function $d : \Sigma_k^* \to [0, \infty)$ satisfying

$$d(w) = k^{-s} \sum_{a \in \Sigma_k} d(wa)$$

for all $w \in \Sigma_k^*$.

Definition. Let $s \in [0, \infty)$ and d be an s-gale. We say that d *succeeds* on a sequence $S \in \Sigma_k^\infty$ if

$$\limsup_{n \to \infty} d(S[0 \dots n]) = \infty.$$

The success set of d is

$$S^\infty[d] = \{S \in \Sigma_k^\infty \mid d \text{ succeeds on } S\}.$$

Definition. We say that a function $d : \Sigma_k^* \to [0,\infty)$ is *p-computable* if there is a function $\hat{d} : \Sigma_k^* \times \mathbb{N} \to \mathbb{Q}$ such that $\hat{d}(w,r)$ is computable in time polynomial in $|w|+r$ and $|\hat{d}(w,r) - d(w)| \leq 2^{-r}$ holds for all w and r.

We say that a function $d : \Sigma_k^* \to [0,\infty) \cap \mathbb{Q}$ is *exactly p-computable* if $d(w)$ is computable in time polynomial in $|w|$.

Definition. Let $X \subseteq \Sigma_k^\infty$, The *p-dimension* of X is

$$\dim_{\mathrm{p}}^{(k)}(X) = \inf \left\{ s \in [0,\infty) \;\middle|\; \begin{array}{l} \text{there is a p-computable } s\text{-gale } d \text{ s.t.} \\ X \subseteq S^\infty[d] \end{array} \right\}$$

By the exact computation lemma in [17] p-computable and exactly p-computable gales are interchangeable in the definition above.

Theorem 2.1 *Let $X \subseteq \Sigma_k^\infty$,*

$$\dim_{\mathrm{p}}^{(k)}(X) = \inf \left\{ s \in [0,\infty) \;\middle|\; \begin{array}{l} \textit{there is an exactly p-computable } s\textit{-gale } d \textit{ s.t.} \\ X \subseteq S^\infty[d] \end{array} \right\}.$$

For a complete introduction and motivation of effective dimension see [19].

2.2 Representations of Reals

We will use infinite sequences over Σ_k to represent real numbers in [0,1). For this, we associate each string $w \in \Sigma_k^*$ with the half-open interval $[w]_k$ defined by $[w]_k = [x, x+k^{-|w|})$, for $x = \sum_{i=1}^{|w|} w[i-1]k^{-i}$. Each real number $\alpha \in [0,1)$ is then represented by the unique sequence $S_k(\alpha) \in \Sigma_k^\infty$ satisfying

$$w \sqsubseteq S_k(\alpha) \iff \alpha \in [w]_k$$

for all $w \in \Sigma_k^*$. We have

$$\alpha = \sum_{i=1}^{\infty} S_k(\alpha)[i-1]k^{-i}$$

and the mapping $\alpha \mapsto S_k(\alpha)$ is a bijection from $[0,1)$ to Σ_k^∞ (notice that $[w]_k$ being half-open prevents double representations). If $x \in \Sigma_k^\infty$ then $\mathrm{real}_k(x) = \alpha$ such that $x = S_k(\alpha)$. Therefore we always have that $\mathrm{real}_k(S_k(\alpha)) = \alpha$ and $S_k(\mathrm{real}_k(x)) = x$. A set of real numbers $A \subseteq [0,1)$ is represented by the set

$$X_k(A) = \{S_k(\alpha) \mid \alpha \in A\}$$

of sequences. If $X \subseteq \Sigma_k^\infty$ then

$$\mathrm{real}_k(X) = \{\mathrm{real}_k(x) \mid x \in X\}.$$

The question addressed in this paper is the following. Is the feasible dimension of a set $A \subseteq [0,1)$ invariant with respect to the base used for representation? That is, if we define $\dim_{\mathrm{p}}(A) = \dim_{\mathrm{p}}^{(k)}(X_k(A))$, is this definition robust when k changes?

2.3 Normality and finite-state dimension

We next include Borel's definition of normal and absolutely normal number [4].

Definition. A real number α is *normal in a base* $k \geq 2$ if, for every $m \geq 1$ and every $w \in \Sigma_k^m$, the following holds

$$\lim_n \frac{|\{i < n \,|S_k(\alpha)[i-m+1..i] = w\}|}{n} = k^{-m}.$$

That is, the asymptotic, empirical frequency of w in the base-k expansion of α is k^{-m}.

Definition. A real number α is *absolutely normal* if it is normal in every base $k \geq 2$.

Finite-state dimension is defined in [8].

Definition. A *finite-state gambler* is a 5-tuple $G = (Q, \delta, \beta, q_0, c_0)$, where

- Q is a nonempty, finite set of *states*,
- $\delta : Q \times \Sigma_k \to Q$ is the *transition function*,
- $\beta : Q \times \Sigma_k \to \mathbb{Q} \cap [0,1]$, with the property $\sum\limits_{i \in \Sigma_k} \beta(q,i) = 1$ for every $q \in Q$ is the *betting function*,
- $q_0 \in Q$ is the *initial state*, and
- c_0, the *initial capital*, is a nonnegative rational number.

Definition. Let $s \in [0, \infty) \cap \mathbb{Q}$ and d be an s-gale. d is *finite-state computable* if there is a finite-state gambler G such that d is the s-gale computed by G, that is, if $G = (Q, \delta, \beta, q_0, c_0)$, then

$$\begin{aligned} &d(\lambda) = c_0, \\ &d(wi) = d(w) \cdot k^s \cdot \beta(\delta^*(q_0, w), i) \end{aligned}$$

for all $w \in \Sigma_k^*$ and $i \in \Sigma_k$.

Definition. Let $X \subseteq \Sigma_k^\infty$, The *FS-dimension* of X is

$$\dim_{\mathrm{FS}}^{(k)}(X) = \inf \left\{ s \in [0,\infty) \;\middle|\; \begin{array}{l} \text{there is a finite-state-computable } s\text{-gale } d \text{ s.t.} \\ X \subseteq S^\infty[d] \end{array} \right\}$$

We also consider the finite-state dimension of a single sequence. $x \in \Sigma_k^\infty$, $\dim_{\mathrm{FS}}^{(k)}(\{x\})$.

Schnorr and Stimm [21] proved that normality is exactly the finite-state case of randomness. Then [5] proves that finite-state randomness and dimension one coincide. A real number α then turns out to be normal in base k if and only if $\dim_{\mathrm{FS}}^{(k)}(\{S_k(\alpha)\}) = 1$ [5].

3 Main Theorem

In this section we prove our main result, p-dimension is base invariant. We need the following lemma with the conversion of p-gales on Σ_l to p-gales on Σ_k.

Lemma 3.1 *Let $k, l \geq 2$. For any exactly p-computable s-gale d on Σ_l and rational $s' > s$, there is a p-computable s'-gale d' on Σ_k such that* $\mathrm{real}_l(S^\infty[d]) \subseteq \mathrm{real}_k(S^\infty[d'])$.

Proof.

Let d be an exactly p-computable s-gale on Σ_l, without loss of generality we assume that $d(\lambda)=1$. For any $n\in\mathbb{N}$, we define a function $D_n:\Sigma_k^*\to[0,\infty)$ as follows. Let $m=\lfloor n\log_k l\rfloor$. For any $y\in\Sigma_k^*$, we define

$$D_n(y)=\begin{cases} k^{s'|y|}\left(\displaystyle\sum_{\substack{x\in\Sigma_l^n\\ [x]_l\subseteq[y]_k}} d(x)+\sum_{\substack{x\in\Sigma_l^n\\ [x]_l\not\subseteq[y]_k\\ [x]_l\cap[y]_k\neq\emptyset}}\frac{1}{2}d(x)\right) & \text{if } |y|\le m\\ k^{(s'-1)\cdot(|y|-m)}D_n(y[0..m-1]) & \text{otherwise.}\end{cases}$$

The desired s'-gale d' on Σ_k is then defined by

$$d'(y)=\sum_{n=0}^{\infty}l^{-s'n}D_n(y).$$

The intuition in the definition of D_n and d' is that for $|y|\le m$, $d'(y)$ takes the full value of $d(x)$ for those x for which y is (the beginning of) the base k representation of x, while it only takes $d(x)/2$ for those x for which we still don't know if (an extension of) y will be the base k representation of (an extension of) x. For $|y|>m$ the otherwise case makes D_n an s'-gale by fair splitting.

Claim 1 d' is an s'-gale on Σ_k.

Let $y\in\Sigma_k^{<m}$. For any $x\in\Sigma_l^n$, we have

$$\begin{aligned}[x]_l\subseteq[y]_k \iff\ & (\exists a\in\Sigma_k)\ [x]_l\subseteq[ya]_k\\ & \text{or } (\exists a\in\Sigma_k-\{(k-1)\})\ [x]_l\subseteq[ya]_k\cup[y(a+1)]_k\end{aligned}$$

because $[x]_l$ can intersect at most two of the intervals $[ya]_k$ for $a\in\Sigma_k$. This is because $|[ya]_k|=k^{-|ya|}\ge k^{-m}\ge k^{-n\log_k l}=l^{-n}=|[x]_l|$. For the same reason, we also have

$$[x]_l\not\subseteq[y]_k \text{ and } [x]_l\cap[y]_k\neq\emptyset \iff$$

$$[([x]_l\not\subseteq[y0]_k \text{ and } [x]_l\cap[y0]_k\neq\emptyset \text{ and } [x]_l\cap[y1]_k=\emptyset) \text{ or}$$

$$([x]_l\not\subseteq[y(k-1)]_k \text{ and } [x]_l\cap[y(k-1)]_k\neq\emptyset \text{ and } [x]_l\cap[y(k-2)]_k=\emptyset)]$$

for any $x\in\Sigma_l^n$. By these relationships, we have

$$\begin{aligned}\sum_{\substack{x\in\Sigma_l^n\\ [x]_l\subseteq[y]_k}} d(x) &= \sum_{a\in\Sigma_k}\sum_{\substack{x\in\Sigma_l^n\\ [x]_l\subseteq[ya]_k}} d(x)+\sum_{a\in\Sigma_k-\{(k-1)\}}\ \sum_{\substack{x\in\Sigma_l^n\\ [x]_l\subseteq[ya]_k\cup[y(a+1)]_k\\ [x]_l\not\subseteq[ya]_k\\ [x]_l\not\subseteq[y(a+1)]_k}} d(x)\\ &= \sum_{a\in\Sigma_k}\sum_{\substack{x\in\Sigma_l^n\\ [x]_l\subseteq[ya]_k}} d(x)+\sum_{\substack{x\in\Sigma_l^n\\ [x]_l\cap[y0]_k\neq\emptyset\\ [x]_l\cap[y1]_k\neq\emptyset}}\frac{1}{2}d(x)\\ &\quad+\sum_{a\in\Sigma_k-\{0,(k-1)\}}\ \sum_{\substack{x\in\Sigma_l^n\\ [x]_l\not\subseteq[ya]_k\\ [x]_l\cap[ya]_k\neq\emptyset}}\frac{1}{2}d(x)+\sum_{\substack{x\in\Sigma_l^n\\ [x]_l\cap[y(k-1)]_k\neq\emptyset\\ [x]_l\cap[y(k-2)]_k\neq\emptyset}}\frac{1}{2}d(x)\end{aligned}$$

this last equality follows by splitting $[x]_l \not\subseteq [ya]_k$ into the two cases $[x]_l \subseteq [ya]_k \cup [y(a+1)]_k$ and $[x]_l \subseteq [y(a-1)]_k \cup [ya]_k$ (for $a \in \Sigma_k - \{0, (k-1)\}$).

We also have that

$$\sum_{\substack{x \in \Sigma_l^n \\ [x]_l \not\subseteq [y]_k \\ [x]_l \cap [y]_k \neq \emptyset}} \frac{1}{2} d(x) = \sum_{\substack{x \in \Sigma_l^n \\ [x]_l \not\subseteq [y0]_k \\ [x]_l \cap [y0]_k \neq \emptyset \\ [x]_l \cap [y1]_k = \emptyset}} \frac{1}{2} d(x) + \sum_{\substack{x \in \Sigma_l^n \\ [x]_l \not\subseteq [y(k-1)]_k \\ [x]_l \cap [y(k-1)]_k \neq \emptyset \\ [x]_l \cap [y(k-2)]_k = \emptyset}} \frac{1}{2} d(x).$$

Combining these two sums establishes that D_n is an s'-gale on Σ_k as follows. Let $y \in \Sigma_k^{<m}$.

$$\begin{aligned}
k^{-s'(|y|+1)} \sum_{a \in \Sigma_k} D_n(ya) &= \sum_{a \in \Sigma_k} \left(\sum_{\substack{x \in \Sigma_l^n \\ [x]_l \subseteq [ya]_k}} d(x) + \sum_{\substack{x \in \Sigma_l^n \\ [x]_l \not\subseteq [ya]_k \\ [x]_l \cap [ya]_k \neq \emptyset}} \frac{1}{2} d(x) \right) \\
&= \sum_{a \in \Sigma_k} \sum_{\substack{x \in \Sigma_l^n \\ [x]_l \subseteq [ya]_k}} d(x) + \sum_{a \in \Sigma_k} \sum_{\substack{x \in \Sigma_l^n \\ [x]_l \not\subseteq [ya]_k \\ [x]_l \cap [ya]_k \neq \emptyset}} \frac{1}{2} d(x) \\
&= \sum_{a \in \Sigma_k} \sum_{\substack{x \in \Sigma_l^n \\ [x]_l \subseteq [ya]_k}} d(x) + \sum_{\substack{x \in \Sigma_l^n \\ [x]_l \not\subseteq [y0]_k \\ [x]_l \cap [y0]_k \neq \emptyset}} \frac{1}{2} d(x) \\
&\quad + \sum_{a \in \Sigma_k - \{0,(k-1)\}} \sum_{\substack{x \in \Sigma_l^n \\ [x]_l \not\subseteq [ya]_k \\ [x]_l \cap [ya]_k \neq \emptyset}} \frac{1}{2} d(x) + \sum_{\substack{x \in \Sigma_l^n \\ [x]_l \not\subseteq [y(k-1)]_k \\ [x]_l \cap [y(k-1)]_k \neq \emptyset}} \frac{1}{2} d(x) \\
&= k^{-s'|y|} \left(\sum_{\substack{x \in \Sigma_l^n \\ [x]_l \subseteq [y]_k}} d(x) + \sum_{\substack{x \in \Sigma_l^n \\ [x]_l \not\subseteq [y]_k \\ [x]_l \cap [y]_k \neq \emptyset}} \frac{1}{2} d(x) \right) \\
&= k^{-s'|y|} D_n(y).
\end{aligned}$$

d' is an infinite sum of s'-gales and is therefore an s'-gale.

Claim 2 d' is p-computable.

We first show how to efficiently compute $D_n(y)$. For this, we iteratively define a sequence of sets $B_i^n(y)$ for $i = 0, \ldots, n$ by

$$B_i^n(y) = \left\{ x \in \Sigma_l^i \,\middle|\, [x]_l \subseteq [y]_k \text{ and } x \text{ has no prefix in } \bigcup_{j=0}^{i-1} B_j^n(y) \right\}.$$

That is, $B_i^n(y)$ is the set of strings x of length i that represent maximal intervals included in $[y]_k$. We can now represent $D_n(y)$ for $y \in \Sigma_k^{\leq m}$ in the following form.

$$D_n(y) = k^{s'|y|} \left(\sum_{i=0}^{n} \sum_{x \in B_i^n(y)} l^{s(n-i)} d(x) + \sum_{\substack{x \in \Sigma_l^n \\ [x]_l \not\subseteq [y]_k \\ [x]_l \cap [y]_k \neq \emptyset}} \frac{1}{2} d(x) \right)$$

This is equivalent to the original definition of D_n because d is an s-gale and therefore

$$\sum_{|u|=n-i} d(xu) = l^{s(n-i)} d(x).$$

Each $B_i^n(y)$ will have at most $2(l-1)$ strings, and these are easily computable by prefix extension. There are two strings to consider for the second sum, the (length $|n|$) predecesor of the first and the successor of the last strings in $\bigcup_i B_i^n(y)$.

For the p-computation of d', let b be such that for every r,

$$\sum_{n=br+1}^{\infty} l^{-n(s'-s)/2} \leq 2^{-r}.$$

Let c be such that $k^{s'} \leq l^{(s'-s)c/2}$. Let

$$f(y,r) = \sum_{n=0}^{br+c|y|} l^{-s'n} D_n(y).$$

Then f is clearly computable in polynomial time on $|y|$ and r and we have that

$$\begin{aligned}
|d'(y) - f(y,r)| &= \sum_{n=br+c|y|+1}^{\infty} l^{-s'n} D_n(y) \\
&\leq \sum_{n=br+c|y|+1}^{\infty} l^{-s'n} k^{s'|y|} D_n(\lambda) \\
&= \sum_{n=br+c|y|+1}^{\infty} l^{-s'n} l^{sn} d(\lambda) k^{s'|y|} \\
&\leq \sum_{n=br+c|y|+1}^{\infty} l^{-(s'-s)n} l^{|y|(s'-s)c/2} \\
&\leq \sum_{n=br+c|y|+1}^{\infty} l^{-(s'-s)n} l^{n(s'-s)/2} \\
&= \sum_{n=br+c|y|+1}^{\infty} l^{-(s'-s)n/2} \\
&\leq 2^{-r}.
\end{aligned}$$

The second and third inequalities come from the fact that both d and D_n are gales, and $d(\lambda) = 1$. Then the fourth one comes from the choice of c and the last one from the choice of b.

Notice that the computation of d' on an input of length n is $O(n^2)$ times the computation time of d on inputs of length $O(n)$.

Claim 3 $\mathrm{real}_l(S^\infty[d])) \subseteq \mathrm{real}_k(S^\infty[d'])$.

Let $\alpha \in [0, 1)$. Letting $x_n = S_l(\alpha)[0..n-1]$ and $y_n = S_k(\alpha)[0..m-1]$, we have $[y_n]_k \cap [x_n]_l \neq \emptyset$ and by definition $D_n(y_n) \geq k^{s'|y_n|}\frac{1}{2}d(x_n)$.

Therefore if $\alpha \in \mathrm{real}_l(S^\infty[d]))$ then $S_l(\alpha) \in S^\infty[d]$ and since for every n,

$$d'(y_n) \geq l^{-s'n} D_n(y_n) \geq l^{-s'n} k^{s'|y_n|}\frac{1}{2}d(x_n) \geq \frac{1}{2}d(x_n),$$

we have that $S_k(\alpha) \in S^\infty[d']$. □

We now have our main theorem.

Theorem 3.2 *For any $A \subseteq [0, 1)$ and $k, l \geq 2$, $\dim_{\mathrm{p}}^{(k)}(X_k(A)) = \dim_{\mathrm{p}}^{(l)}(X_l(A))$.*

Proof. Let $s > \dim_{\mathrm{p}}^{(l)}(X_l(A))$. By Theorem 2.1 there is an exactly p-computable s-gale d on Σ_l such that $X_l(A) \subseteq S^\infty[d]$ thus $A \subseteq \mathrm{real}_l(S^\infty[d])$. For each $s' > s$ the previous lemma gives a p-computable s'-gale d' on Σ_k such that $\mathrm{real}_l(S^\infty[d])) \subseteq \mathrm{real}_k(S^\infty[d'])$. Therefore $X_k(A) \subseteq S^\infty[d']$, and $\dim_{\mathrm{p}}^{(k)}(X_k(A)) \leq s'$. As $s > \dim_{\mathrm{p}}^{(l)}(X_l(A))$ and $s' > s$ were arbitrary, this establishes $\dim_{\mathrm{p}}^{(k)}(X_k(A)) \leq \dim_{\mathrm{p}}^{(l)}(X_l(A))$. The converse inequality follows by a symmetric argument. □

This contrasts with the fact that finite-state dimension [8] is not invariant under base change (consequence of [5, 21]). For instance, sequences with finite-state dimension 1 coincide with normal sequences [5, 21], and normality is not invariant under base change. Thus we prove that any real number for which polynomial-time dimension is 1 in a any base is absolutely normal.

Corollary 3.3 *For any $x \in [0, 1)$ and $k \geq 2$, if $\dim_{\mathrm{p}}^{(k)}(\{X_k(x)\}) = 1$ then x is absolutely normal.*

Proof. It is simple to see that p-dimension 1 implies finite-state dimension 1, therefore if

$$\dim_{\mathrm{p}}^{(k)}(\{X_k(x)\}) = 1$$

then $X_k(x)$ has finite-state dimension 1 and therefore x is normal in base k. By our main theorem $\dim_{\mathrm{p}}^{(l)}(\{X_l(x)\}) = 1$ holds for every $l \geq 2$ and x is normal in every base l. □

Bienvenu [3] observed that Corollary 3.3 can be used to prove the following corollary after reading an early version [12] of the present paper. Previously Ki and Linton [14] had shown that for each fixed base $k \geq 2$, the class of $\mathcal{N}_k$ of numbers normal to base k is $\mathbf{\Pi}_3^0$-complete.

Corollary 3.4 (Bienvenu) *The class $\mathcal{N}$ of absolutely normal numbers is $\mathbf{\Pi}_3^0$-complete.*

Proof. Let $\mathcal{D}_0$ be the class of density 0 numbers, i.e. $z \in \mathcal{D}_0$ iff

$$\lim_{n\to\infty} \frac{|\{i < n \mid S_2(z)[i] = 1\}|}{n} = 0.$$

Then $\mathcal{D}_0$ is $\mathbf{\Pi}_3^0$-complete [14] under Wadge reductions, that is, for any $\mathbf{\Pi}_3^0$ class $\mathcal{C}$, there is a continuous function f such that

$$z \in \mathcal{C} \iff f(z) \in \mathcal{D}_0.$$

Now fix a computable real β of p-dimension 1. By Corollary 3.3, β is absolutely normal.

For any real z, we define a reduction $g(z) = \beta^{f(z)}$ as follows. Define

$$S_2(\beta^{f(z)})[i] = \begin{cases} \beta[i] & \text{if } S_2(f(z))[i] = 0 \\ 0 & \text{if } S_2(f(z))[i] = 1 \end{cases}$$

Then if $z \in \mathcal{C}$, $f(z) \in \mathcal{D}_0$ has density 0, which means β and $\beta^{f(z)}$ have the same p-dimension, so $\beta^{f(z)} \in \mathcal{N}$ is also absolutely normal. On the other hand if $z \notin \mathcal{C}$, then $f(z) \notin \mathcal{D}_0$ has positive density causing $\dim_{\mathrm{p}}(\beta^{f(z)}) < 1$, so $\beta^{f(z)}$ is not normal base two and hence not absolutely normal. Therefore g is a Wadge reduction of $\mathcal{C}$ to $\mathcal{N}$. □

We note that the proof of Corollary 3.4 does not require our result on p-dimension and could have in principle been proven earlier using the base invariance of pspace-dimension or constructive dimension, for example.

Acknowledgment

We thank anonymous referees for their valuable comments.

References

[1] K. B. Athreya, J. M. Hitchcock, J. H. Lutz, and E. Mayordomo. Effective strong dimension in algorithmic information and computational complexity. *SIAM Journal on Computing*, 37(3):671–705, 2007.

[2] J. L. Balcázar, J. Díaz, and J. Gabarró. *Structural Complexity I.* Springer-Verlag, Berlin, second edition, 1995.

[3] L. Bienvenu. Personal communication, 2013.

[4] É. Borel. Sur les probabilités dénombrables et leurs applications arithmétiques. *Rendiconti del Circolo Matematico di Palermo*, 27(1):247–271, 1909.

[5] C. Bourke, J. M. Hitchcock, and N. V. Vinodchandran. Entropy rates and finite-state dimension. *Theoretical Computer Science*, 349(3):392–406, 2005.

[6] V. Brattka, J. S. Miller, and A. Nies. Randomness and differentiability. Technical Report arXiv:1104.4465v3 [math.LO], arXiv.org, 2011.

[7] C. Calude. *Information and Randomness: An Algorithmic Perspective,* Second Edition. Springer-Verlag, Berlin, 2002.

[8] J. J. Dai, J. I. Lathrop, J. H. Lutz, and E. Mayordomo. Finite-state dimension. *Theoretical Computer Science*, 310:1–33, 2004.

[9] R. G. Downey and D. R. Hirschfeldt. *Algorithmic randomness and complexity.* Springer-Verlag, 2010.

[10] J. M. Hitchcock. Fractal dimension and logarithmic loss unpredictability. *Theoretical Computer Science*, 304(1–3):431–441, 2003.

[11] J. M. Hitchcock, J. H. Lutz, and E. Mayordomo. The fractal geometry of complexity classes. *SIGACT News Complexity Theory Column*, 36:24–38, 2005.

[12] J. M. Hitchcock and E. Mayordomo. Base invariance of feasible dimension. Isaac Newton Institute Preprint NI12052-SAS, 2012.

[13] J. M. Hitchcock and N. V. Vinodchandran. Dimension, entropy rates, and compression. *Journal of Computer and System Sciences*, 72(4):760–782, 2006.

[14] H. Ki and T. Linton. Normal numbers and subsets of $\mathbb{N}$ with given densities. *Fundamenta Mathematicae*, 144:163–179, 1994.

[15] M. López-Valdés and E. Mayordomo. Dimension is compression. *Theory of Computing Systems*, 52(1):95–112, 2013.

[16] J. H. Lutz. Almost everywhere high nonuniform complexity. *Journal of Computer and System Sciences*, 44(2):220–258, 1992.

[17] J. H. Lutz. Dimension in complexity classes. *SIAM Journal on Computing*, 32(5):1236–1259, 2003.

[18] J. H. Lutz. The dimensions of individual strings and sequences. *Information and Computation*, 187(1):49–79, 2003.

[19] J. H. Lutz. Effective fractal dimensions. *Mathematical Logic Quarterly*, 51(1):62–72, 2005.

[20] W. M. Schmidt. On normal numbers. *Pacific J. Math*, 10(2):661–672, 1960.

[21] C.-P. Schnorr and H. Stimm. Endliche automaten und zufallsfolgen. *Acta Informatica*, 1(4):345–359, 1972.

[22] J. Silveira. Invariancia por cambio de base de la aleatoriedad computable y la aleatoriedad con recursos acotados. Master's thesis, Universidad de Buenos Aires, 2011.